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Ultrasonic wave propagation in a functionally graded thick cylinder

Asghar Najafi[a*]

[a] *Research Assistant Professor, Mechanical Rotary Equipment Research Dept., Niroo Research Institute, at the end of Shahid Dadman Blvd., Shahrak Ghods, 1468613113, Tehran, Iran.*

* *Corresponding author e-mail: anajafi@nri.ac.ir*

Abstract

In this paper, wave propagation in thick-walled and long cylindrical pressure vessels made of functionally graded material (FGM) is investigated. Material properties are considered as a power function of radius and the Poisson's ratio is assumed as constant. The equation of motion which is a wave equation is obtained and solved analytically. The equation is a Sturm Liouville problem. By an appropriate change of variable, it can be turned into a simple Bessel differential equation. The Eigenvalue equation is obtained based on boundary conditions and using this equation natural frequencies are calculated. A mathematical expression for the mean velocity of radial stress wave propagation is purposed. Wave propagation produced by a dynamic internal pressure is simulated and the results are discussed. The radial and hoop stress are obtained for different material distributions. The mean wave velocity relation with inhomogeneity constants is studied. Finally, the analytical results are verified by finite element methods.

Keywords: Wave propagation; Functionally graded; Thick cylinder; Sturm Liouville equation

1. Introduction

Nowadays, by the ability to produces material with variation in their properties, great advances have been made in design of shells and vessels. An FGM is non-homogeneous in composition, so all of its properties such as modulus of elasticity, Poisson's ratio, density and yield limit may or less vary throughout the material. This exceptional property is ideal for designers and manufacturers to produce low weight and strong parts. Knowing this, many researches studies static and dynamic behaviors of vessels and shells made of FGM.

Obata and Noda (1994) investigated the thermal stress in an FGM hollow sphere and in a hollow circular cylinder using a perturbation approach [1]. The aim of their study was to understand the effect of composition on stresses and to design the optimum FGM hollow circular cylinder and hollow sphere. Assuming that the material has a graded modulus of elasticity, while the Poisson's ratio is a constant, Tutuncu and Ozturk (2001) investigated the stress distribution in axisymmeric structures [2]. They obtained the closed-form solutions for stresses and displacements in functionally graded cylindrical and spherical vessels under internal pressure.

Rahimi and Zamani Nejad investigated stress and displacement distribution in rotating thick hollow FGM cylinders [3]. They found a closed form solution for the problem. In a general case, the field equations for shells of revolutions were derived by Zamaninejad et. al.[4].

Chen et. al. investigated free vibration of thick orthotropic piezoelectric hollow cylinder which was filled of a non-viscous compressible fluid based on state space formulation. They presented an analytical frequency equation for different boundary condition at inner and outer cylindrical surface [5].

Transient waves in FGM hollow cylinders were studied by Hen et. al. using a hybrid numerical method [6]. Shakeri et. al. studied wave propagation in a thick hollow cylinder made of FGM [7]. They considered that the cylinder is divided into many thin layers and assumed that each layer is isotope. They obtained time history of radial stress along cylinder thickness using a numerical method (finite element analysis). Hosseini studied thermoelasticity behavior as well as the second sound and elastic wave propagation in FGM thick hollow cylinders [8]. He assumed mechanical properties of cylinder are graded across the thickness as a power law function of radius and considered isotropic multi layered thin cylinders for finite difference analysis.

In this paper, equation of motion for a thick hollow and infinite long cylinder made of FGM is derived and the solution is obtained. It is assumed that physical properties of the material vary by a power function of radius.

2. Modelling and equations of motion

In this section, the equations of motion for vibration and propagation of waves in a thick cylinder will be derived. Suppose a thick cylinder with internal radius r_i and external radius r_o which is under an internal pressure *P*. Considering plane strain condition, one can obtain equilibrium equation in cylindrical coordinate as follows:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = \rho \frac{\partial^2 u}{\partial t^2} \tag{1}$$

Where σ_{rr} and $\sigma_{\theta\theta}$ are radial and hoop stress respectively. u is radial displacement and ρ stands for density. Radial and hoop strain can be related as:

$$\varepsilon_r = \frac{\partial u}{\partial r} \tag{2}$$

$$\varepsilon_\theta = u / r \tag{3}$$

Stress-strain relations for nonhomogeneous and isotropic materials are as follows:

$$\sigma_r = (c_1 \varepsilon_r + c_2 \varepsilon_\theta) E \tag{4}$$

$$\sigma_\theta = (c_2 \varepsilon_r + c_1 \varepsilon_\theta) E \tag{5}$$

where, for plane strain condition, c_1 and c_2 are as follows:

$$c_1 = \frac{(1-\upsilon)}{(1+\upsilon)(1-2\upsilon)}, \; c_2 = \frac{\upsilon}{(1+\upsilon)(1-2\upsilon)} \tag{6}$$

One can find the Navier equation in terms of radial displacement, u, using equations 1-5:

$$r^2 \frac{\partial^2 u}{\partial^2 r} + \left(\frac{r^2}{E} \frac{dE}{dr} + r \right) \frac{\partial u}{\partial r} + \left(\frac{nr}{E} \frac{dE}{dr} - 1 \right) u = \rho \frac{c_1}{E} r^2 \frac{\partial^2 u}{\partial t^2} \tag{7}$$

where: $n = \frac{c_2}{c_1} = \upsilon/(1-\upsilon)$

The material is assumed to be dependent on radial coordinate by a power function. If radius normalized by $\bar{r} = r/r_i$ then module of elasticity and density are:

$$E = E_i \bar{r}^{\beta} \tag{8}$$

$$\rho = \rho_i \bar{r}^{\alpha} \tag{9}$$

where E_i and ρ_i are the module of elasticity and density at inner wall $r = r_i$. α and β are inhomogeneity constants determined experimentally.

Since for the most of engineering materials variation of Poisson ratio, υ, is small and the current analysis is performed for thick wall cylinders this parameter is considered as a constant.

The boundary condition for inner and outer wall of the cylinder is as follow:

For the inner wall:

$$\sigma_r = F(t) \tag{10}$$

For the outer wall:

$$\sigma_r = 0 \tag{11}$$

By putting (7) and (8) in (6) following equation will result:

$$r^2 \frac{\partial^2 u}{\partial^2 r} + (1+\beta) r \frac{\partial u}{\partial r} + (n\beta - 1) u = \gamma r^{2+\alpha-\beta} \frac{\partial^2 u}{\partial t^2} \tag{12}$$

where $\gamma = \frac{\rho_i r_i^{\alpha-\beta}}{E_i c_1}$. This equation can be turned into a Sturm Liouville problem. By multiplying the equation by $r^{\beta-1}$, one has:

$$r^{1+\beta} \frac{\partial^2 u}{\partial^2 r} + (1+\beta) r^{\beta} \frac{\partial u}{\partial r} + (n\beta - 1) r^{\beta-1} u = \gamma r^{1+\alpha} \frac{\partial^2 u}{\partial t^2} \tag{13}$$

This equation can be written as follow:

$$\frac{\partial}{\partial r}\left[p(r) \frac{\partial u}{\partial r} \right] - q(r) u = s(r) \frac{\partial^2 u}{\partial^2 t} \tag{14}$$

where: $p(r) = r^{\beta+1}$, $q(r) = (n\beta - 1) r^{\beta-1}$ and $s(r) = \gamma r^{1+\alpha}$

Equation (13) is an Sturm Liouville problem [8]. Kernel integral for the equation can be obtained as follows:

$$\frac{d}{dr}\left[p(r) \frac{dK}{dr} \right] - q(r) K = -\lambda^2 s(r) K \tag{15}$$

By putting *s*, *p* and *q* in the above equation, one has:

$$r^2 K'' + (1+\beta) r K' + \left(n\beta - 1 + \lambda^2 \gamma r^{2+\alpha-\beta}\right) K = 0 \tag{16}$$

where $v = -\beta/2$, $a = \frac{2+\alpha-\beta}{2}$, $c = \frac{\sqrt{4 - 4n\beta + \beta^2}}{2+\alpha-\beta}$ and $b = \sqrt{\gamma}/a$. Following change of variable is applied to this equation:

$$W = r^{-v} K \tag{17}$$

$$z = \lambda b r^{a} \tag{18}$$

Following equation will result:

$$z^2 \frac{d^2 W}{dz^2} + z \frac{dW}{dz} + \left(z^2 - c^2\right) W = 0 \tag{19}$$

which is a simple Bessel differential equation. The solution for this equation can be written as $W = AJ_c(z) + BY_c(z)$. By putting *z* and *W* from equation (16) and (17), kernel integral is obtained:

$$K = r^v\left[AJ_c\left(\lambda br^a\right) + BY_c\left(\lambda br^a\right)\right] \tag{20}$$

Boundary condition for kernel integral is similar to the original boundary condition in equations (9) and (10). However, for kernel integral, the boundary condition should be homogeneous as follows:

$$K' + nK/r\big|_{r=r_i} = 0 \tag{21}$$

$$K' + nK/r\big|_{r=r_o} = 0 \tag{22}$$

By substituting *K* from equation (19) in (20) and (21) these equations will result:

$$\left[axJ_c'(x) + (n+v)J_c(x)\right]A + \left[axY_c'(x) + (n+v)Y_c(x)\right]B = 0 \tag{23}$$

$$\left[a\eta xJ_c'(\eta x) + (n+v)J_c(\eta x)\right]A + \left[a\eta xY_c'(\eta x) + (n+v)Y_c(\eta x)\right]B = 0 \tag{24}$$

Where $x = \lambda br_i^a$ and $\eta = (r_0/r_i)^a$. In order to have non-trivial solution for equation (22) and (23), determinant of the coefficient must be equal to zero:

$$\begin{aligned}&\left[axJ_c'(x) + (n+v)J_c(x)\right]\times\left[a\eta xY_c'(\eta x) + (n+v)Y_c(\eta x)\right] - \\ &-\left[axY_c'(x) + (n+v)Y_c(x)\right]\times\left[a\eta xJ_c'(\eta x) + (n+v)J_c(\eta x)\right] = 0\end{aligned} \tag{25}$$

Solving this equation numerically, x and corresponding eigenvalue can be found. If x_m be the m’th roots of equation (24), m’th eigenvalue can be obtained as:

$$\lambda_m = x_m / br_i^a \tag{26}$$

In this way natural frequency of thick hollow FGM cylinder can be estimated:

$$f_m = \lambda_m / 2\pi \tag{27}$$

Considering equation (22), A and B for each eigenvalue are:

$$\begin{aligned}A(\lambda_m) &= \lambda abr_i^a Y_c'\left(\lambda br_i^a\right) + (n+v)Y_c\left(\lambda br_i^a\right)\\ B(\lambda_m) &= \lambda abr_i^a J_c'\left(\lambda br_i^a\right) + (n+v)J_c\left(\lambda br_i^a\right)\end{aligned} \tag{28}$$

Normal kernel integral (normal mode shape) can be found as:

$$\begin{aligned}k(\lambda_m, r) &= K(\lambda_m, r)/\|K_m\|\\ \|K_m\|^2 &= \int_{r_i}^{r_o} s(r)K(\lambda_m, r)^2\,dr\end{aligned} \tag{29}$$

Substituting K from equation (19) into the above equation, one has:

$$\|K_m\|^2 = \int_{r_i}^{r_o}\gamma r^{1+\alpha}\left\{r^v\left[AJ_c\left(\lambda br^a\right) + BY_c\left(\lambda br^a\right)\right]\right\}^2 dr = \int_{r_i}^{r_o}\gamma r^{2a-1}\left[AJ_c\left(\lambda br^a\right) + BY_c\left(\lambda br^a\right)\right]^2 dr \tag{30}$$

The following change of variable is applied to this equation:

$$\chi = r^a \tag{31}$$

So the following equation will result:

$$\|K_m\|^2 = \gamma/a\int_{r_i^a}^{r_o^a}\chi\left[AJ_c(\lambda b\chi) + BY_c(\lambda b\chi)\right]^2 d\chi \tag{32}$$

Analytical evaluation of the above integral is [7]:

$$\|K_m\|^2 = \gamma\chi^2/2a\left\{\left[AJ_c'(\lambda b\chi) + AY_c'(\lambda b\chi)\right]^2 + \left(1 - \frac{c^2}{\lambda^2 b^2\chi^2}\right)\left[AJ_c(\lambda b\chi) + AY_c(\lambda b\chi)\right]^2\right\}\Bigg|_{r_i^a}^{r_o^a} \tag{33}$$

where $\tilde{u}(\lambda, t) = \int_{r_i}^{r_o} u(r,t)s(r)k(\lambda, r)dr$

By multiplying equation (11) by $s(r)k(\lambda, r)$ and integrating by part following equation will result:

$$f(t) - g(t) - \lambda^2\tilde{u} = \ddot{\tilde{u}} \tag{34}$$

where $f(t)=\left[k(\lambda,r)p(r)\frac{\partial u}{\partial r}\right]_{r=r_i}^{r=r_o}$ and $g(t)=\left[up(r)\frac{dk}{dr}\right]_{r=r_i}^{r=r_o}$. From boundary conditions in (20) and (21), one can write:

$$k'+nk/r\big|_{r=r_i}=0 \text{ and } k'+nk/r\big|_{r=r_o}=0 \tag{35}$$

Using these equation and equation (4) following equation will result:

$$\ddot{\tilde{u}}+\lambda_m^2\tilde{u}=p(r)k(\lambda_m,r)\sigma_r/c_1E(r)\big|_{r_i}^{r_o} \tag{36}$$

Considering boundary conditions in (9) and (10) following equation can be obtained:

$$\ddot{\tilde{u}}+\lambda_m^2\tilde{u}=p(r_i)k(\lambda_m,r_i)F(t)/c_1E_i \tag{37}$$

Final solution is:

$$u(r,t)=\sum_{n=1}^{\infty}\tilde{u}(\lambda_m,t)k(\lambda_m,r) \tag{38}$$

3. Mean velocity of wave propagation

Transferring all terms in equation (11) into the left side yields:

$$\gamma r^{2+\alpha-\beta}\frac{\partial^2 u}{\partial t^2}-r^2\frac{\partial^2 u}{\partial^2 r}-(1+\beta)r\frac{\partial u}{\partial r}-(n\beta-1)u=0 \tag{39}$$

This equation is in the form of a wave equation and corresponding characteristic equation is:

$$\frac{dr}{dt}=\sqrt{r^2/\gamma r^{2+\alpha-\beta}}=r^{\frac{\beta-\alpha}{2}}/\sqrt{\gamma} \tag{40}$$

The time needed for passing wave between r_1 and r_2 is $\Delta t=b\left(r_2^a-r_1^a\right)$. Mean value of velocity is:

$$\bar{V}=\frac{r_2-r_1}{b\left(r_2^a-r_1^a\right)} \tag{41}$$

4. Numerical results

Consider a thick hollowed cylinder with internal radius $r_i=25mm$ and external radius $r_o=50mm$. The module of elasticity and density at internal surface are $E_i=200Gpa$ and $\rho_i=8000kg/m^3$ respectively.

Natural frequencies for this example are calculated for various values of β and compared by a finite element analysis (Table 1). A good agreement is observed. Fig. 1 shows first natural frequency versus β for different values of α. By increasing β and decreasing α first natural frequency will go up. This is completely expected since stiffness is increased by β and increasing α reduces the density.

Table 1. Natural frequencies for FGM cylinder, Finite element analysis and present study results

β=2	PS	2715.3	14654	28337	42234	56185	70158
	FEM	2768.1	14380	28198	42144	56124	70120
β=1	PS	2501.5	13196	25742	38433	51160	63901
	FEM	2576.7	13019	25652	38376	51123	63882
β=0	PS	2338.7	11879	23338	34895	46475	58065
	FEM	2432.9	11783	23291	34867	46459	58061
β=-1	PS	2218.1	10691	21119	31614	42123	52637
	FEM	2327.5	10662	21108	31610	42125	52646

β=-2	PS	2131.5	9621	19075	28579	38091	47605
	FEM	2252.3	9644.9	19095	28595	38108	47627

Now consider following time dependent load at internal surface:

$$F(t) = P_i t/\tau \text{ for } t \le \tau \text{ and } F(t) = 0 \text{ for } t > \tau \tag{39}$$

For $t \le \tau$ equation (27) is as follow:

$$\ddot{\tilde{u}} + \lambda_m^2 \tilde{u} = \mu p(r_i) k(\lambda_m, r_i) t / c_1 E_i = \kappa t \tag{40}$$

where $\kappa = P_i p(r_i) k(\lambda_m, r_i) / r_i E_i \tau$. The solution for this equation is straight forward:

$$\tilde{u} = \kappa t / \lambda_m^2 + C_1 \sin \lambda_m t + C_2 \cos \lambda_m t \tag{41}$$

In which, for specified initial condition $C_2 = 0$ and $C_1 = -\kappa / \lambda_m^3$. For $t \le \tau$ equation (27) become $\ddot{\tilde{u}} + \lambda_m^2 \tilde{u} = 0$. The solution for this equation is $\tilde{u} = C_3 \sin(t-\tau) + C_4 \cos(t-\tau)$. To have a continuous response at $t = \tau$, one must has:

$$\begin{aligned} C_3 &= \gamma\tau / \lambda_m^2 + C_1 \sin \lambda_m \tau = \gamma(\lambda_m \tau - \sin \lambda_m \tau) / \lambda_m^3 \\ C_4 &= \gamma / \lambda_m^2 + C_1 \lambda_m \cos \lambda_m \tau = \gamma(1 - \cos \lambda_m \tau) / \lambda_m^2 \end{aligned} \tag{40}$$

Suppose $P_i = 80Mpa$ and $\tau = 0.005s$. In Fig. 1 time history of displacement is shown for different values of β. By applying load, displacement will go up linearly and after unloading it oscillates harmonically. By increasing β which causes higher stiffness, the amplitude of vibration reduces. Hoop and radial stresses at middle point of thickness is shown in Fig's. 2 and 3. It is clear from Fig. 2 that the frequency and the rate of change of stress and motion of wave is higher for larger values of β. This is completely in agreement with this fact that for higher ratio of stiffness to density, one can expect faster wave propagation. Using time simulation, one can find approximate value of mean velocity and compare with equation (29). Fig. 4 shows the mean velocity of stress wave versus β. By increasing β the mean velocity of wave will increase monotonically due to this fact that stiffness increases by β.

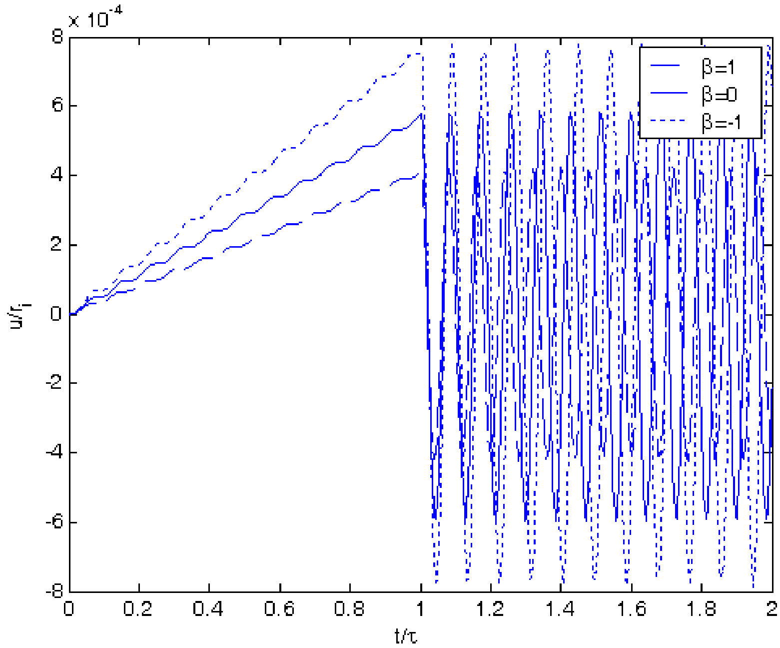

Fig. 1. Time history of displacement for different values of β

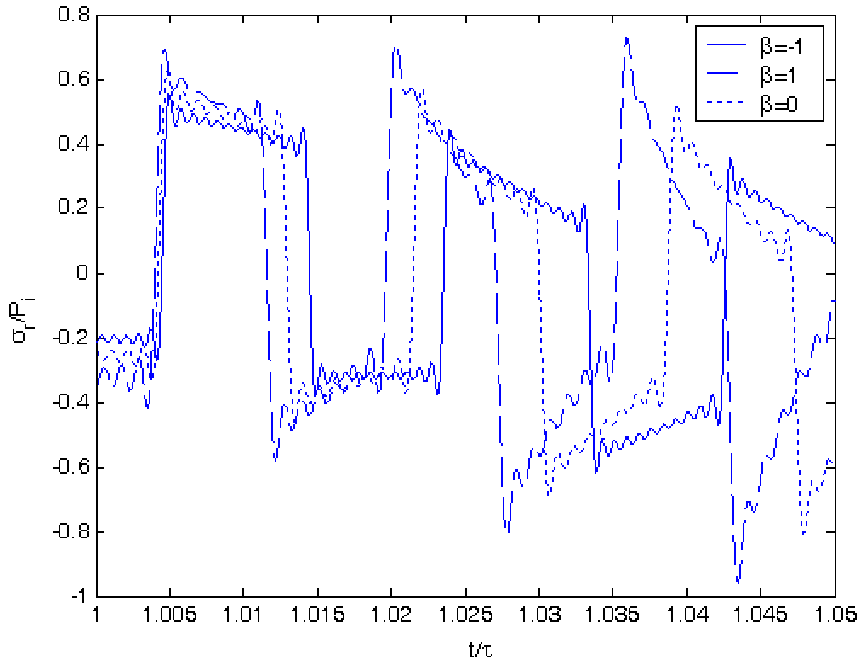

Fig. 2 Time history of radial stress at middle thickness for tree value of β

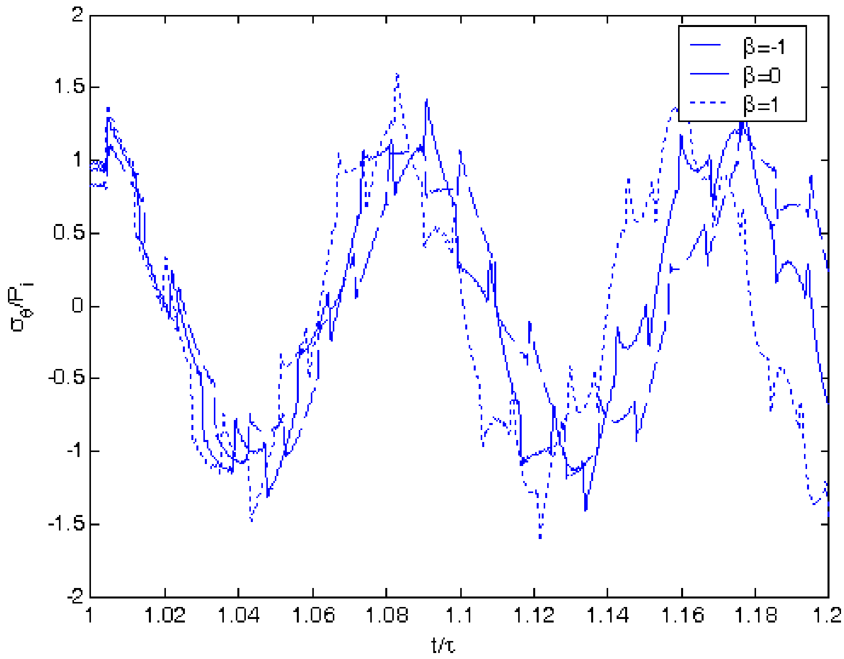

Fig .3. Time history of hoop stress at middle thickness for tree value of β

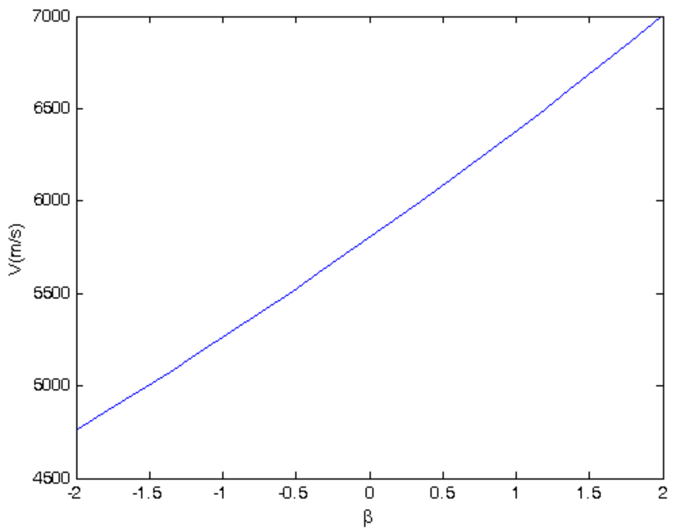

Fig. 4. Mean value of velocity of wave propagation versus β

5. Conclusion

For the case that Material properties are varied as a function of the radius of the cylinder to a power function, Equation of motion which is a wave equation is obtained and solved analytically. The equation is a sturm liouville problem and by an appropriate change of variable can be turned into a simple Bessel differential equation. The Eigenvalue equation is obtained based on boundary conditions and using this equation natural frequencies are calculated. An expression for mean velocity of radial stress wave propagation is purposed.

A comparison between FEM analysis and the proposed solution shows a good agreement. In addition to this, at steady state condition others analytical methods in literatures validate the solution. Based on this solution, wave propagation produced by a dynamic internal pressure is simulated and the results are discussed. Mean velocity of stress wave are obtained both by time history of responses and characteristic equation.

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